

# Development of a Cox-Thompson inverse scattering method to charged particles

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## Abstract

Cox-Thompson fixed-energy quantum inverse scattering method is developed further to treat long range Coulomb interaction. Depending on the reference potentials chosen, two methods have been formulated which produce inverse potentials with singular or finite value at the origin. Based on the quality of reproduction of input experimental phase shifts, it is guessed that the  $p - \alpha$  interaction possesses an interesting repulsive hard core.

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## 1 Introduction

Inverse quantum scattering methods [1, 2, 3, 4] represent useful techniques to assess effective interactions between colliding composite particles in a model independent way. In nuclear physics the modified Newton-Sabatier (mNS) method [5, 6] has been used to determine, in general, complex valued optical potentials describing various systems, e.g.,  $^{12}\text{C} - ^{12}\text{C}$  elastic scattering [7, 8] or to guess the spin-orbit potentials arising in the  $p - \alpha$  and  $n - \alpha$  collisions [9]. The Newton-Sabatier (NS) method [10, 11] has the property that it requires an infinite set of phase shifts otherwise the first moment of the potential generated vanishes. This induces an unpleasant oscillation of the potential at large distances, in addition to the artificial singularity at the origin which remains to be present in the potentials generated also by the mNS method.

In the recent decade another procedure, the Cox-Thompson (CT) inverse quantum scattering method [12] has been investigated [13, 14, 15]. This method has the advantage that it requires a finite set of phase shifts and possesses a non-zero first momentum of the potential generated. In addition to this, in general, it also reproduces test potentials better than the mNS method [13]. The CT method has been applied so far to uncharged particle scattering in order to reconstruct short ranged potentials from synthetic phase shifts [13], and also to construct  $n - \alpha$  potential from experimental data [14]. We shall develop here the method further with the intention that the CT method can be applied to charged particle scattering when the potential to be constructed has a long ranged Coulomb tail. The various extensions will be applied to both synthetic and experimental phase shifts. The results are obtained by using the nonlinear solver package of MATLAB.

## 2 Cox-Thompson (CT) method

The original CT method belongs to the fixed-energy procedures when a finite set of  $N$  phase shifts  $\{\delta_l\}_{l \in S}$  is converted to the spherical potential  $V(r)$  appearing in the  $l$ -th partial wave radial Schrödinger equation

$$\left( -\frac{\hbar^2}{2m} \frac{d^2}{dr^2} + \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2} + V(r) - \frac{\hbar^2}{2m} k^2 \right) \Psi_l(k; r) = 0 \quad (1)$$

with boundary conditions  $\Psi_l(k; 0) \propto r^{l+1}$  and  $\Psi_l(k; r \rightarrow \infty) \propto \sin(kr - l\frac{\pi}{2} + \delta_l)$ .

We shall also refer to another form of the Schrödinger equation

$$x^2 \left( \frac{d^2}{dx^2} + 1 - q(x) \right) \psi_l(x) = l(l+1) \psi_l(x), \quad (2)$$

where dimensionless quantities have been introduced for the distance  $x = kr$ , for the potential  $q(x) = E^{-1}V(xk^{-1})$  with  $E = \hbar^2 k^2 / (2m)$ , and the dimensionless wave function  $\psi_l(x)$  is related to the wave function  $\Psi_l(k; r)$  through the asymptotic normalization constant  $\Psi_l(k; r) = C_l \psi_l(kr)$ .

The conversion from phase shifts  $\{\delta_l\}$  to potential  $V(r)$  can be carried out in two steps [13, 14]. First, we solve either of the following two systems of highly non-linear equations

$$e^{2i\delta_l} = \frac{1 + i\mathcal{K}_l^+}{1 - i\mathcal{K}_l^-} \quad \text{or} \quad \tan(\delta_l) = \frac{\mathcal{K}_l^+ + \mathcal{K}_l^-}{2 + i(\mathcal{K}_l^+ - \mathcal{K}_l^-)}, \quad l \in S, \quad (3)$$

with

$$\mathcal{K}_l^\pm = \sum_{L \in T, l' \in S} [M_{\sin}]_{lL} [M_{\cos}^{-1}]_{Ll'} e^{\pm i(l-l')\pi/2}, \quad l \in S. \quad (4)$$

and

$$\left\{ \begin{array}{c} M_{\sin} \\ M_{\cos} \end{array} \right\}_{lL} = \frac{1}{L(L+1) - l(l+1)} \left\{ \begin{array}{c} \sin \left( (l-L)\frac{\pi}{2} \right) \\ \cos \left( (l-L)\frac{\pi}{2} \right) \end{array} \right\}, \quad l \in S, L \in T \quad (5)$$

for the set  $T$  of the shifted angular momenta  $\{L\}$  where the relations  $S \cap T = \emptyset$  and  $|T| = |S| = N$  hold. Next, one calculates the potential as

$$V(r) = Eq(kr), \quad (6)$$

where

$$q(x) = -\frac{2}{x} \frac{d}{dx} \left( \frac{K(x, x)}{x} \right) \quad (7)$$

with

$$K(x, y) = \sum_{L \in T} A_L(x) u_L(y), \quad (8)$$

where the coefficient functions  $A_L(x)$  are calculated by solving the system of linear equations

$$\sum_{L \in T} A_L(x) \frac{W[u_L(x), v_l(x)]}{l(l+1) - L(L+1)} = v_l(x), \quad l \in S \quad (9)$$

with

$$u_n(x) = \sqrt{\frac{\pi x}{2}} J_{n+\frac{1}{2}}(x), \quad v_n(x) = \sqrt{\frac{\pi x}{2}} Y_{n+\frac{1}{2}}(x), \quad n \in S \cup T \quad (10)$$

being the regular and irregular solutions of the dimensionless free radial Schrödinger equation for the  $n$ th partial wave (Riccati-Bessel functions [16]). Note that the set  $T$  of shifted angular momenta  $L$  contains real or (in case of complex phase shifts) even complex numbers.

Using the behaviour of these functions at the origin and infinity, one easily derives the following properties of the Cox-Thompson potential. At the origin we get [13], in general, a constant

$$q(0) = Q - 2(1 - Q) \sum_{L \in T} \sum_{l \in S} \frac{G_{lL}^{-1}}{2l - 1}. \quad (11)$$

with the matrix  $G_{Ll} = (L - l)^{-1}$  and the constant quantity  $Q = \sum_{L \in T} \sum_{l \in S} \frac{G_{lL}^{-1}}{L + \frac{3}{2}}$ .

At the infinity the CT potential in general falls off like an inverse power of two

$$q(x \rightarrow \infty) = -\frac{1}{x^2} (\alpha_T \cos(2x) - \beta_T \sin(2x)) + \frac{2}{x^3} (\alpha_T \sin(2x) + \beta_T \cos(2x) + \gamma_T) \quad (12)$$

where the coefficients depending on the set  $T$  are defined by the following relations

$$\alpha_T = \frac{1}{2} \sum_{L \in T} \left( a_L \cos L\frac{\pi}{2} - b_L \sin L\frac{\pi}{2} \right), \quad (13)$$

$$\beta_T = -\frac{1}{2} \sum_{L \in T} \left( b_L \cos L\frac{\pi}{2} + a_L \sin L\frac{\pi}{2} \right), \quad (14)$$

$$\gamma_T = \frac{1}{2} \sum_{L \in T} \left( b_L \cos L\frac{\pi}{2} - a_L \sin L\frac{\pi}{2} \right), \quad (15)$$

with the coefficients  $a_L, b_L$  defined via the asymptotic expansion functions  $A_L^a(x) \equiv A_L(x \rightarrow \infty)$  which admits a periodic expansion [15] as

$$A_L^a(x) = a_L \cos(x) + b_L \sin(x) \quad (16)$$

because of the form of the asymptotic version of equation (9)

$$\sum_{L \in T} A_L^a(x) \frac{\cos((l-L)\frac{\pi}{2})}{l(l+1) - L(L+1)} = -\cos(x - l\frac{\pi}{2}), \quad l \in S. \quad (17)$$

### 3 Extensions of CT method to long ranged potential

In nuclear physics one frequently encounters charged particle scattering. The appearance of the long ranged Coulomb potential requires an appropriate extension of the CT method which will be carried out in various ways in this section. All the different extensions rely upon the two-potential formalism [17].

Consider a spherical potential  $V(r)$  which can be written as a sum of a short-ranged interior (or nuclear) part  $\hat{V}$  and a long-ranged exterior (asymptotic or reference) part  $V^{(0)}(r)$ , i.e.,

$$V(r) = \hat{V}(r) + V^{(0)}(r). \quad (18)$$

Then the phase shift itself is also split into two parts

$$\delta_l = \hat{\delta}_l + \delta_l^{(0)} \quad (19)$$

where  $\delta_l^{(0)}$  is the phase shift due to  $V^{(0)}$  and  $\hat{\delta}_l$  corresponds to the phase shifts caused by  $\hat{V}$  (in the presence of  $V^{(0)}$ ). If, for instance,  $\hat{V}(r) \equiv 0$ ,  $\hat{\delta}_l = 0 \forall l$ .

If the short ranged part of the potential is zero beyond a finite distance  $r_a$ ,

$$\hat{V}(r) = 0, \quad r \geq r_a \quad (20)$$

then, accordingly, the radial wave function can be written in this region as

$$\Psi_l(k; r \geq r_a) = C_l \left( F_l^{(0)}(kr) + \tan \hat{\delta}_l G_l^{(0)}(kr) \right). \quad (21)$$

In the latter equation the functions  $F_l^{(0)}$  and  $G_l^{(0)}$  mean, respectively, the regular and irregular solutions of the dimensionless radial Schrödinger equation (2) with potential  $q(x) = E^{-1}V^{(0)}(xk^{-1})$ .

#### 3.1 Phase transformation method (PCT)

We may design a modified potential  $\tilde{V}(r)$  which is identical with the original one (up to a constant energy shift) within the interior region  $0 < r < r_a$  and zero outside

$$\tilde{V}(r) = \begin{cases} V(r) - V(r_a), & r \leq r_a, \\ 0, & r \geq r_a. \end{cases} \quad (22)$$

Thus if equation (20) applies the  $\hat{V}$  interior potential can be deduced from  $\tilde{V}$ . The radial scattering wave function of this problem at the shifted energy  $E_B = E - V(r_a) \equiv \hbar^2 k_B^2 / 2m$  is identical with the original one in the inside region

$$\tilde{\Psi}_l(k_B; r) = \Psi_l(k; r), \quad r \leq r_a \quad (23)$$

but differs in the outside region

$$\tilde{\Psi}_l(k_B; r) = \tilde{C}_l \left( u_l(k_B r) - \tan \tilde{\delta}_l v_l(k_B r) \right), \quad r \geq r_a. \quad (24)$$

The equality of the logarithmic derivatives at  $r = r_a$

$$\left. \frac{d}{dr} \log \left( \frac{1}{r} \tilde{\Psi}_l(k_B; r) \right) \right|_{r=r_a} = \left. \frac{d}{dr} \log \left( \frac{1}{r} \Psi_l(k; r) \right) \right|_{r=r_a} \quad (25)$$

with  $\Psi_l(k; r)$  from (21) gives the transformed phase shifts  $\tilde{\delta}_l(k_B)$  which can be used to (re)construct the short ranged potential  $\tilde{V}(r)$  (22). To get the whole original potential one simply adds the reference potential

$$\tilde{V}^{(0)}(r) = \begin{cases} V(r_a), & r \leq r_a, \\ V^{(0)}(r), & r \geq r_a, \end{cases} \quad (26)$$

if it is known.

The phase shift transformation method enables the use of the CT method as formulated in the preceding section. It requires that the modified potential  $\tilde{V}(r)$  be constructed from the set  $\{\tilde{\delta}_l\}$  by the actual inverse method as accurately as possible. But no exact fixed energy inverse scattering method has yet been formulated which is able to produce a short range potential that is exactly zero beyond a finite radius, if a finite set of phase shifts is employed. Therefore, one always introduces a small error into the potential (re)construction when applying the phase shift transformation procedure.

The phase shift transformation method has been introduced by May, Münchow and Scheid [6] and is widely applied in case of the modified Newton Sabatier method. It is applied also to the CT inverse procedure to treat Coulomb scattering [18] and we shall call this combined method as the PCT method.

### 3.2 Generalized CT scheme

Based on the two potential formalism one can derive a generalized CT (gCT) scheme which employs the given nuclear phase shifts  $\{\hat{\delta}_l\}$  for constructing the short-ranged (nuclear) potential  $\hat{V}(r)$  but avoids use of a matching radius  $r_a$ . The derivation starts with the ansatz for the input symmetrical kernel of the Gel'fand-Levitan-type integral equation

$$g(x, x') = \sum_{l \in S} \gamma_l F_l^{(0)}(x_{<}) G_l^{(0)}(x_{>}), \quad (27)$$

where  $x_{<}(x_{>})$  denotes the lesser (greater) of  $x, x'$ . Then, by proceeding through the usual steps [13, 14], one arrives at a system of nonlinear equations identical in structure to (3)

$$e^{2i\hat{\delta}_l} = \frac{1 + i\mathcal{K}_l^+}{1 - i\mathcal{K}_l^-} \quad \text{or} \quad \tan(\hat{\delta}_l) = \frac{\mathcal{K}_l^+ + \mathcal{K}_l^-}{2 + i(\mathcal{K}_l^+ - \mathcal{K}_l^-)}, \quad l \in S, \quad (28)$$

with

$$\mathcal{K}_l^\pm = \sum_{L \in T, l' \in S} [M_{\sin}]_{lL} [M_{\cos}^{-1}]_{Ll'} e^{\pm i[(l-l')\pi/2 + \delta_{l'}^{(0)} - \delta_l^{(0)}]}, \quad l \in S, \quad (29)$$

and

$$\left\{ \begin{array}{c} M_{\sin} \\ M_{\cos} \end{array} \right\}_{lL} = \frac{1}{L(L+1) - l(l+1)} \left\{ \begin{array}{c} \sin \left( (l-L)\frac{\pi}{2} + \delta_L^{(0)} - \delta_l^{(0)} \right) \\ \cos \left( (l-L)\frac{\pi}{2} + \delta_L^{(0)} - \delta_l^{(0)} \right) \end{array} \right\}, \quad l \in S, L \in T \quad (30)$$

for the set  $T$  of the shifted angular momenta  $L$  where the relations  $S \cap T = \emptyset$  and  $|T| = |S|$  hold. From the set  $T$  one calculates the nuclear potential as

$$\hat{V}(r) = Eq(kr), \quad (31)$$

where

$$q(x) = -\frac{2}{x} \frac{d}{dx} \left( \frac{K(x, x)}{x} \right) \quad (32)$$

with

$$K(x, y) = \sum_{L \in T} A_L(x) F_L^{(0)}(y). \quad (33)$$

The coefficient function  $A_L(x)$  is calculated by solving the system of linear equations

$$\sum_{L \in T} A_L(x) \frac{W[F_L^{(0)}(x), G_l^{(0)}(x)]}{l(l+1) - L(L+1)} = G_l^{(0)}(x), \quad l \in S, \quad (34)$$

From this gCT scheme several developments are possible as the actual form of the reference potential has not yet been specified.

### 3.3 Coulomb reference potential method (CCT)

If one sets the reference potential to be the bare Coulomb potential

$$V^{(0)}(r) = E \frac{2\eta}{kr} \equiv \frac{1}{4\pi\epsilon_0} \frac{Z_1 Z_2 e^2}{r}, \quad (35)$$

with  $\eta$  being the Sommerfeld parameter then one arrives at the Coulomb CT (CCT) method. In this case the regular and irregular reference functions are the regular and irregular Coulomb functions, the reference phase shift becomes the Coulomb phase

$$\sigma_l = \frac{1}{2i} \ln \left[ \frac{\Gamma(l+1+i\eta)}{\Gamma(l+1-i\eta)} \right], \quad l \in S. \quad (36)$$

Here it should be mentioned that in the course of application of CCT method it may become necessary to know the regular and irregular Coulomb functions and Coulomb phases for complex orders. For example, in case of non-elastic scattering the phases are complex and therefore, as noted before, the  $L$  numbers are also complex valued. The Coulomb functions are well-defined for complex orders and similarly to the real order case they can be given as power series (for details see e.g. [19, 20] and Appendix).

It is interesting that contrary to the fact that the asymptotic form of the reference functions contains the well known logarithmic term  $-\eta \ln 2kr$  in the argument, it does not appear in the gCT formulas because of cancellation.

Using the known power series of the Coulomb functions it can be shown that the CCT method gives a potential which is proportional to the Coulomb potential near the origin. For one term, i.e.  $|T| = |S| = 1$ , we get

$$V(r \approx 0) = E \left[ \frac{L(1+l)}{l(1+L)} \right] \frac{2\eta}{kr} + O(1). \quad (37)$$

For large  $r$  we get

$$V(r \rightarrow \infty) = E \frac{2\eta}{kr} - \frac{2E}{(kr)^2} \sum_{L \in T} \sum_{l \in S} [M_{\cos}^{-1}]_{Ll} \cos(\Theta_L(kr) + \Theta_l(kr)) + O\left(\frac{1}{(kr)^3}\right), \quad (38)$$

with  $\Theta_L(x) = x - \eta \ln 2x - L \frac{\pi}{2} + \sigma_L$ .

We see that the CCT method generates an inverse potential that gives a Coulomb-like singularity at the origin and produces a damped oscillation around the Coulomb tail at large distances. It is free of the matching parameter  $r_a$  and requires just the nuclear phase shifts  $\hat{\delta}_l$  which are derived by the usual phase shift analysis procedures.

### 3.4 Modified Coulomb reference potential method (MCT)

In order to obtain an inverse potential that is finite at the origin, instead of being singular there, we can modify the Coulomb reference potential accordingly. This reference potential is constant in the interior domain and purely Coulombic outside. This modified Coulomb potential is the same as that employed by the phase transformation method for Coulomb asymptotics and reads as

$$V^{(0)}(r) = \begin{cases} E \frac{2\eta}{kr_a}, & r \leq r_a, \\ E \frac{2\eta}{kr}, & r \geq r_a \end{cases}. \quad (39)$$

To this reference potential there belong the following regular and irregular radial wave function

$$F_l^{(0)}(kr) = \begin{cases} u_l \left( \sqrt{1 - \frac{2\eta}{kr_a}} \cdot kr \right), & r < r_a, \\ \alpha_{F_l^{(0)}} F_l(kr) + \beta_{F_l^{(0)}} G_l(kr), & r > r_a, \end{cases} \quad (40)$$

$$G_l^{(0)}(kr) = \begin{cases} v_l \left( \sqrt{1 - \frac{2\eta}{kr_a}} \cdot kr \right), & r < r_a, \\ \alpha_{G_l^{(0)}} F_l(kr) + \beta_{G_l^{(0)}} G_l(kr), & r > r_a, \end{cases} \quad (41)$$

and reference phase shift

$$\delta_l^{(0)} = \sigma_l + \arctan \left( \frac{\beta_{F_l^{(0)}}}{\alpha_{F_l^{(0)}}} \right). \quad (42)$$

The coefficients  $\alpha_{F_l^{(0)}}, \beta_{F_l^{(0)}}, \alpha_{G_l^{(0)}}, \beta_{G_l^{(0)}}$  can be calculated from the equality of the inner and outer wave functions and their derivatives at the matching radius  $r_a$ . For example, the two coefficients necessary for calculating the reference phase shift are

$$\alpha_{F_l^{(0)}} = \frac{\frac{u_l(\sqrt{1-\frac{2\eta}{x_a}} \cdot x_a)}{G_l(x_a)} - \frac{\sqrt{1-\frac{2\eta}{x_a}} u'_l(\sqrt{1-\frac{2\eta}{x_a}} \cdot x_a)}{G'_l(x_a)}{\frac{F_l(x_a)}{G_l(x_a)} - \frac{F'_l(x_a)}{G'_l(x_a)}}, \quad \beta_{F_l^{(0)}} = \frac{\frac{u_l(\sqrt{1-\frac{2\eta}{x_a}} \cdot x_a)}{F_l(x_a)} - \frac{\sqrt{1-\frac{2\eta}{x_a}} u'_l(\sqrt{1-\frac{2\eta}{x_a}} \cdot x_a)}{F'_l(x_a)}{\frac{G_l(x_a)}{F_l(x_a)} - \frac{G'_l(x_a)}{F'_l(x_a)}}, \quad (43)$$

with  $x_a = kr_a$  and prime denotes derivation with respect to argument.

The total phase shifts can be written in two different ways

$$\delta_l = \hat{\delta}_l + \sigma_l = \delta_l^{\text{MCT}} + \delta_l^{(0)} \quad (44)$$

where  $\hat{\delta}_l$  means the nuclear phase shifts given as data and  $\delta_l^{\text{MCT}}$  is to be used to perform the CT inverse calculation outlined above.

The potential obtained by the MCT method has a finite value at the origin. Because this method employs a similar reference potential as the PCT method, the results provided by the two methods should also be very similar, although quite different functions are used in the calculations. The advantage of the MCT over the PCT lies in that it does not involve the small error in the phase shift reproduction inherent to the PCT.

## 4 Applications to (re)construct effective potentials

### 4.1 Synthetic phase shifts

To illustrate the general applicability of the long-range CT inversion procedures we shall first use them to reconstruct model potentials. We model the  $\alpha - \alpha$  scattering with two slightly different potentials: the first is finite at the origin and the second is singular (describing a possible non-locality).

The first model potential is given by

$$V(r) = U(r) + V_C(r), \quad (45)$$

where the nuclear interaction is described by a Woods-Saxon form

$$U(r) = U_0 \left( 1 + d \frac{r-R}{a} \right)^{-1}, \quad (46)$$

and the Coulomb interaction is represented by the potential of a homogeneous charged sphere as

$$V_C(r) = \begin{cases} \frac{Z_1 Z_2 e^2}{2R_C} \left( 3 - \frac{r^2}{R_C^2} \right), & r \leq R_C, \\ \frac{Z_1 Z_2 e^2}{r}, & r > R_C. \end{cases} \quad (47)$$

For the various parameters we choose the following values:  $Z_1 = 2, Z_2 = 2, A_1 = 4, A_2 = 4, U_0 = -20$  MeV,  $R = R_C = 2.0636$  fm,  $a = 0.25$  fm.

We have calculated the nuclear phase shifts  $\hat{\delta}_l$  at energies  $E = 25$  and  $35$  MeV (see tables 1 and 2). These phase shifts are then used as input data for the various CT calculations. The results for the  $L$ -values obtained using the nonlinear solver of MATLAB are shown in tables 1 and 2. The corresponding potentials are displayed in figure 1.

One can observe that as expected the PCT and MCT procedures give almost the same results and the CCT inverse potential is divergent at the origin. By increasing the scattering energy more phase shifts become available for the inversion (see tables 1 and 2) and the potential reproduction becomes better.

The second model is obtained by adding the singular potential term

$$V_{\text{sing}}(r) = \frac{d^{-r}}{r^2} \quad (48)$$

to the previous model, i.e.,

$$V(r) = U(r) + V_C(r) + V_{\text{sing}}(r). \quad (49)$$

The reconstruction of phases is listed in table 3 and the potentials are shown in figure 2. We see that the CCT potential follows nicely the model potential in the singular domain near the origin while the PCT and MCT methods are unable to reproduce the singularity although their phase shift reconstruction is good.

Table 1: Model data ( $\delta_l^{\text{orig}}$  phase shifts), inversion results ( $L$  shifted angular momenta and  $\Delta\delta_l$  differences between the model phase shifts and the ones given by the various methods $\ddagger$ ) of the  $\alpha - \alpha$  scattering at  $E_{\text{c.m.}} = 25$  MeV. The matching parameter used in the PCT and MCT procedures was set to  $r_a = 10$  fm.

$l$	$L^{\text{CCT}}$	$L^{\text{MCT}}$	$L^{\text{PCT}}$	$\delta_l^{\text{orig}}$	$\Delta\delta_l^{\text{CCT}}$	$\Delta\delta_l^{\text{MCT}}$	$\Delta\delta_l^{\text{PCT}}$
0	-1.5622	-0.9225	-0.9219	1.2989	0.0085	0.0053	0.0056
1	0.5286	0.5841	0.5843	1.1445	0.0011	0.0009	0.0001
2	1.6462	1.7503	1.7502	0.8307	0.0116	0.0063	0.0062
3	2.9468	3.0281	3.0281	0.2300	0.0131	0.0025	0.0024
4	4.0456	4.1114	4.1115	0.0399	0.0176	0.0053	0.0054
5	5.0272	5.0983	5.0982	0.0062	0.0080	0.0019	0.0018
6	6.0308	6.0874	6.0875	0.0009	0.0073	0.0004	0.0005
7	7.0169	7.0684	7.0684	0.0001	0.0047	0.0039	0.0039
8	8.0177	8.0557	8.0557	0.0000	0.0001	0.0066	0.0066
9	9.0108	9.0428	9.0427	0.0000	0.0001	0.0108	0.0108

$\ddagger$  Note that for the sake of comparison the phases given by the inverse potentials were calculated by cutting-off the non-physical oscillations beyond the matching radius  $r_a$  used in the PCT procedure. Without the cut-off the MCT and CCT potentials reproduce the phase shifts within an error of the numerical precision.

Table 2: Model data and inversion results of the  $\alpha - \alpha$  scattering at  $E_{\text{c.m.}} = 35$  MeV. The matching parameter used in the PCT and MCT procedures was set to  $r_a = 10$  fm.

$l$	$L^{\text{CCT}}$	$L^{\text{MCT}}$	$L^{\text{PCT}}$	$\delta_l^{\text{orig}}$	$\Delta\delta_l^{\text{CCT}}$	$\Delta\delta_l^{\text{MCT}}$	$\Delta\delta_l^{\text{PCT}}$
0	-1.4866	-0.8389	-0.8361	1.1880	0.0032	0.0002	0.0033
1	0.4954	0.6158	0.6161	0.9864	0.0023	0.0011	0.0001
2	1.6375	1.7077	1.7076	0.8542	0.0032	0.0027	0.0027
3	2.8420	2.9144	2.9144	0.4246	0.0069	0.0031	0.0033
4	4.0002	4.0677	4.0677	0.1111	0.0030	0.0017	0.0016
5	5.0256	5.0873	5.0873	0.0231	0.0079	0.0043	0.0044
6	6.0277	6.0873	6.0873	0.0043	0.0001	0.0002	0.0002
7	7.0199	7.0713	7.0713	0.0008	0.0032	0.0019	0.0019
8	8.0176	8.0627	8.0627	0.0001	0.0047	0.0029	0.0030
9	9.0124	9.0492	9.0492	0.0000	0.0011	0.0009	0.0009
10	10.0115	10.0421	10.0420	0.0000	0.0034	0.0030	0.0031
11	11.0083	11.0315	11.0316	0.0000	0.0051	0.0044	0.0044
12	12.0082	12.0267	12.0267	0.0000	0.0007	0.0067	0.0067

## 4.2 Experimental phase shifts

Our goal is to assess the effective central potential governing the  $p - \alpha$  scattering events. Comprehensive data of phase shift analysis of the  $p - \alpha$  scattering has been presented by Ali, Ahmad and Ferdous in [21]. Because of the spin-orbit coupling both spin-up  $\delta_l^+$  and spin-down  $\delta_l^-$  phase shifts contribute to the scattering amplitude at each partial wave. In case of weak spin-orbit coupling the combined phase shifts

$$\hat{\delta}_l = \frac{1}{2l+1}[(l+1)\delta_l^+ + l\delta_l^-] \quad (50)$$

are characteristic of the underlying central potential [22], and we shall use them as input for the CT procedures.

We have inverted the phase shift data of Ali *et al.* Since the inverse potentials exhibit similar characteristics we present only some of the inversion results here.

In figure 3 the inverse potentials yielded by three CT methods at  $E_{\text{lab}} = 17.45$  MeV proton energy are depicted. The results at this energy are representative of the potentials recovered below the  $E_{\text{lab}} = 22.94$  MeV,  $\alpha + p \rightarrow d + {}^3\text{He}$  inelastic threshold. As we see the MCT and PCT potentials are almost identical and strongly resemble a Woods-Saxon form. The range and strength of all the three potentials are similar

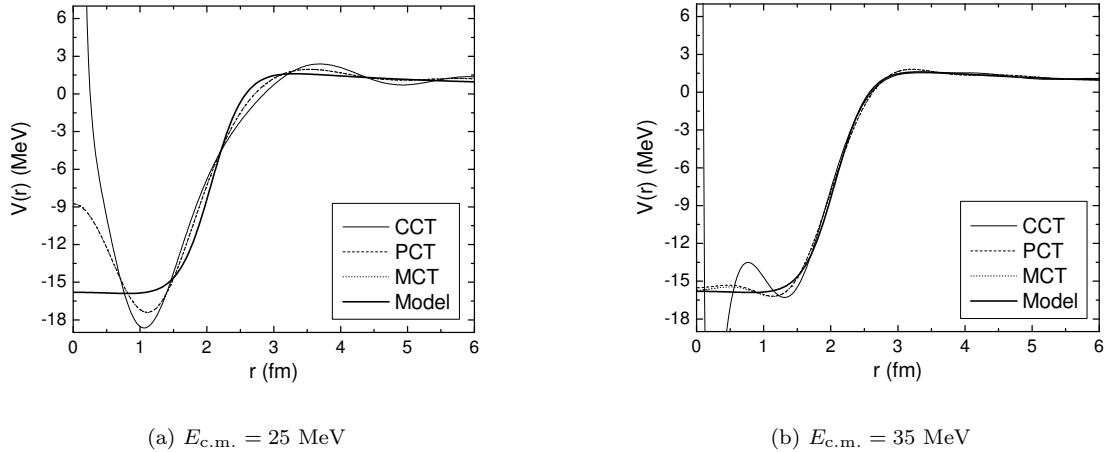


Figure 1: Model with potential (45) and inverse potentials yielded by the CCT, MCT and PCT methods (labeled accordingly) at  $E_{c.m.} = 25$  and  $35$  MeV center of mass energies. The model potential is non-singular at the origin.

but the CCT potential is different in shape: it possesses a repulsive core.

Note that a repulsion at small distances can be theoretically accounted for as the manifestation of the Coulombic non-locality [23]. It can also be seen that this repulsion core stabilizes the inverse potential in the sense that the amplitude of the asymptotic oscillations is diminished compared to the non-repulsive MCT/PCT results. This is also the reason why the phase shift reproduction with a given precision of the CCT potential is better than that of the MCT/PCT potentials (see table ??).

In figure 4 numerous other CCT and PCT potentials are shown at various energy values including those above inelastic threshold. (The MCT potentials are not shown because they coincide within the width of line with the PCT results.) Apart from the Coulombic singularity at the origin, the potentials have a similar range of  $3 - 4$  fm and strength of  $50 - 70$  MeV (PCT) and  $50 - 160$  MeV (CCT). The imaginary part is much less compared to the real part. The reproduction of phase shifts (not shown) gets better at higher energy in case of the PCT potentials because of the fixed cut-off radius which in principle does not apply to CCT (and MCT) method. Without use of this radius the CCT (and MCT) potentials give back the input phase shifts exactly.

## 5 Conclusions, summary

We have developed the Cox-Thompson (CT) fixed energy inverse quantum scattering method into various directions in order to make it appropriate for treating long range interactions. By explicit calculation we have shown that the modified (MCT) and phase transformed (PCT) methods yield practically the same potentials although quite other functions are involved in the calculation. These potentials are generally finite at the origin therefore they can be used for cases with no singularity at small distances. The Coulomb (CCT) method produces a characteristic Coulomb singularity at the origin. This method can be applied with success when nonlocality or a repulsion hard core plays a role in the interaction.

We have applied the methods first to model cases with and without singularity. The model potentials describing  $\alpha - \alpha$  scattering are well reproduced by the new methods according to their characteristic properties concerning the finiteness or infiniteness at the origin. Then, the experimental  $p - \alpha$  phases of Ali *et al* [21] have been inverted with the result that the CCT potentials showing up a repulsive hard core reproduce better the input phase shifts than the PCT potentials which exhibit a Woods-Saxon shape. Whether or not the repulsive core is present between the proton and alpha particle when being scattered by each other should be clarified also using independent source of information (see e.g. [21, 24, 25, 26, 27]).

## Acknowledgements

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Table 3: Model data and inversion results for the divergent model potential at  $E_{\text{c.m.}} = 25$  MeV. The matching parameter used in the PCT and MCT procedures was set to  $r_a = 10$  fm.

$l$	$L^{\text{CCT}}$	$L^{\text{MCT}}$	$L^{\text{PCT}}$	$\delta_l^{\text{orig}}$	$\Delta\delta_l^{\text{CCT}}$	$\Delta\delta_l^{\text{MCT}}$	$\Delta\delta_l^{\text{PCT}}$
0	-1.5257	-0.8526	-0.8522	1.2121	0.0046	0.0038	0.0041
1	0.5116	0.5862	0.5862	1.1314	0.0011	0.0003	0.0007
2	1.6606	1.7506	1.7505	0.8245	0.0051	0.0043	0.0042
3	2.9505	3.0290	3.0290	0.2284	0.0061	0.0028	0.0027
4	4.0401	4.1108	4.1109	0.0395	0.0094	0.0060	0.0060
5	5.0293	5.0978	5.0977	0.0060	0.0036	0.0013	0.0013
6	6.0283	6.0870	6.0871	0.0008	0.0040	0.0031	0.0031
7	7.0179	7.0677	7.0677	0.0001	0.0030	0.0023	0.0023
8	8.0163	8.0553	8.0553	0.0000	0.0001	0.0008	0.0008
9	9.0113	9.0413	9.0413	0.0000	0.0020	0.0020	0.0020
10	10.0105	10.0325	10.0325	0.0000	0.0031	0.0024	0.0025
11	11.0077	11.0224	11.0224	0.0000	0.0018	0.0019	0.0019
12	12.0073	12.0155	12.0155	0.0000	0.0008	0.0014	0.0014
13	13.0056	13.0093	13.0093	0.0000	0.0005	0.0010	0.0010
14	14.0053	14.0065	14.0066	0.0000	0.0003	0.0006	0.0006
15	15.0043	15.0044	15.0043	0.0000	0.0002	0.0003	0.0003

Table 4: Inversion results of the experimental  $p - \alpha$  data at  $E_{\text{lab}} = 17.45$  MeV. The matching parameter used in the PCT and MCT procedures was set to  $r_a = 7$  fm from which distance also the MCT and CCT potentials have been replaced by the pure Coulombic tail.

$l$	$L^{\text{CCT}}$	$L^{\text{MCT}}$	$L^{\text{PCT}}$	$\delta_l^{\text{orig}}$	$\Delta\delta_l^{\text{CCT}}$	$\Delta\delta_l^{\text{MCT}}$	$\Delta\delta_l^{\text{PCT}}$
0	-1.7204	-1.6703	-1.6705	1.7240	0.0139	0.0219	0.0198
1	0.5252	0.4537	0.4534	1.4839	0.0249	0.0589	0.0597
2	2.0328	2.0210	2.0211	0.0760	0.0206	0.0107	0.0108
3	3.0475	3.0881	3.0882	0.0229	0.0250	0.0762	0.0765

## A Coulomb functions of complex order

In certain applications of both the CCT and MCT formulations of the Cox-Thompson inverse scattering method for long-ranged potentials it is necessary to evaluate the regular and irregular Coulomb wave functions for complex orders. Based on [19, 20] the evaluation is accomplished by using the power series given below.

The regular Coulomb wave function for  $L \in \mathbb{C}$  complex order is given by

$$F_L(x) = \frac{2^L d^{-\frac{\pi\eta}{2}} [\Gamma(L+1+i\eta)\Gamma(L+1-i\eta)]^{\frac{1}{2}}}{\Gamma(2L+2)} \cdot x^{L+1} \sum_{j=L+1}^{\infty} C_j^L(\eta) x^{j-L-1}, \quad (51)$$

and the  $C_j^L$  constants are defined by the recursion

$$C_{L+1}^L = 1, \quad (52)$$

$$C_{L+2}^L = \frac{\eta}{L+1}, \quad (53)$$

$$C_j^L = \frac{2\eta C_{j-1}^L - C_{j-2}^L}{(j+L)(j-L-1)} \quad j > L+2. \quad (54)$$

This production of the  $F_L(x)$  is a simple analytic continuation of the formulae in [16].

However the irregular Coulomb wave function cannot be given by such a simple generalization. We utilize therefore the fact that the Coulomb wave functions and the Whittaker functions are related to each other linearly for fixed  $L$  and  $\eta$ . By means of the Whittaker functions the regular Coulomb wave

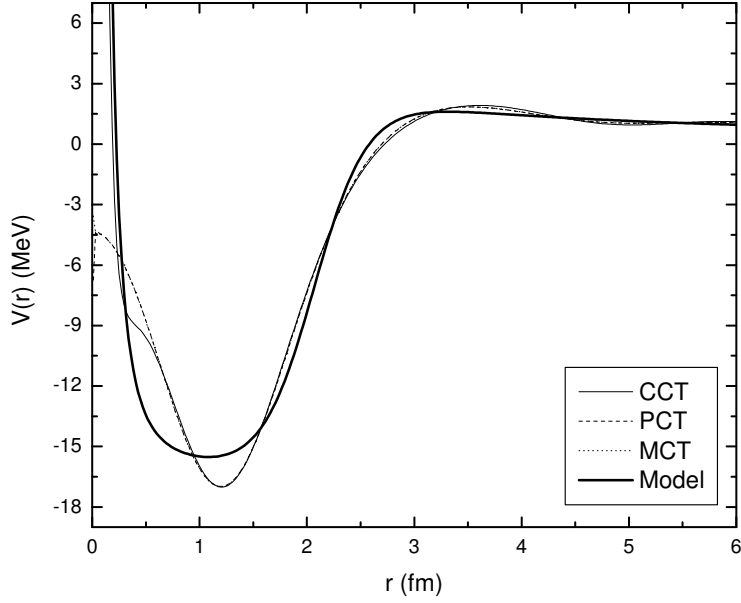


Figure 2: Model with potential (49) and inverse potentials yielded by the CCT, MCT and PCT methods (labeled accordingly) at  $E_{\text{c.m.}} = 25$  MeV center of mass energy. The model potential is singular at the origin.

function is also given.

$$F_L(x) = A(L, \eta) M\left(i\eta, L + \frac{1}{2}, 2ix\right), \quad (55)$$

$$G_L(x) = iA(L, \eta) M\left(i\eta, L + \frac{1}{2}, 2ix\right) + B(L, \eta) W\left(i\eta, L + \frac{1}{2}, 2ix\right) \quad (56)$$

with

$$A(L, \eta) = \frac{d^{-i(L+1)\frac{\pi}{2}} d^{-\frac{\pi\eta}{2}} [\Gamma(L+1+i\eta)\Gamma(L+1-i\eta)]^{\frac{1}{2}}}{2\Gamma(2L+2)} \quad (57)$$

$$B(L, \eta) = \frac{d^{+iL\frac{\pi}{2}} d^{+\frac{\pi\eta}{2}} [\Gamma(L+1+i\eta)\Gamma(L+1-i\eta)]^{\frac{1}{2}}}{\Gamma(L+1+i\eta)}. \quad (58)$$

The Whittaker functions  $M(a, b, y)$  and  $W(a, b, y)$  are well-known for complex arguments (see [16] chapter 13). The Gamma function was calculated by a Lanczos series approximation.

The  $\eta$  parameter appearing in the above formulae is the Sommerfeld parameter given with the quantities discussed earlier as

$$\eta = \frac{k}{E} \frac{e^2}{8\pi\epsilon_0} Z_1 Z_2 = \frac{\sqrt{2}}{\hbar} \frac{e^2}{8\pi\epsilon_0} \cdot \sqrt{\frac{m}{E}} Z_1 Z_2. \quad (59)$$

## References

- [1] Chadan K and Sabatier P C 1977 *Inverse Problems in Quantum Scattering Theory* (New York: Springer Verlag)
- [2] von Geramb H (ed) 1994 *Quantum Inversion Theory and Applications (Lecture Notes in Physics vol 427)* (Berlin: Springer)
- [3] Apagyi B, Endrédi G and Lévy P (eds) 1997 *Inverse and Algebraic Quantum Scattering Theory (Lecture Notes in Physics vol 488)* (Berlin: Springer)
- [4] Apagyi B and Horváth M (eds) 2008 *Proceedings of the International Conference on Inverse Quantum Scattering Theory (Special Issue, Modern Physics Letters B vol 22, issue 23)* (World Scientific)

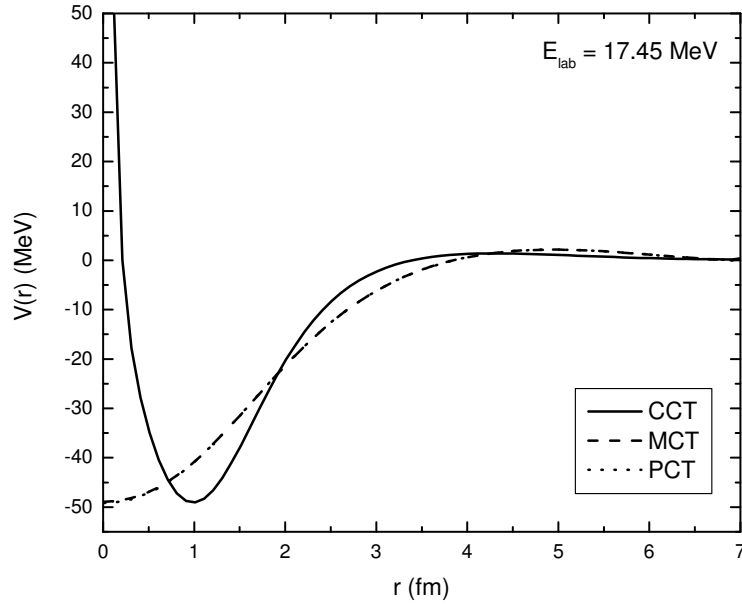
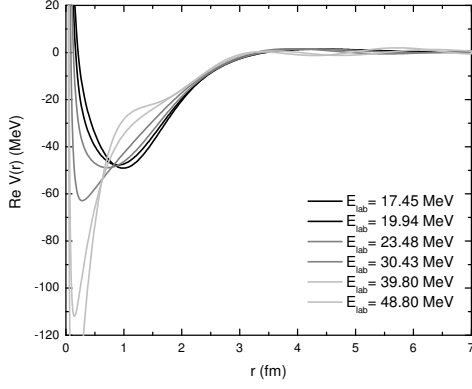
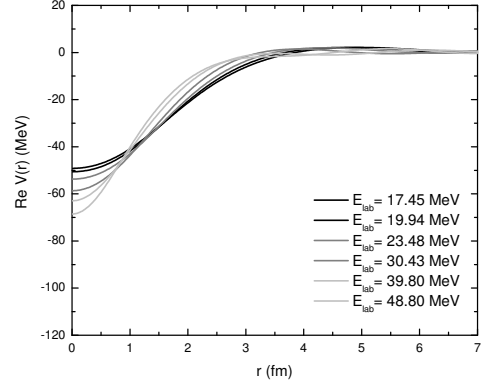


Figure 3: Inverse  $p - \alpha$  potentials  $V(r)$  obtained from input phase shifts  $\hat{\delta}_l^{\text{orig}}$  (given in table ??) as a function of the radial distance  $r$  at energy  $E_{\text{lab}} = 17.45$  MeV ( $E_{\text{c.m.}} = 13.96$  MeV,  $k = 0.731$  fm $^{-1}$ ). Curves obtained by the PCT, MCT and CCT method are labeled accordingly.

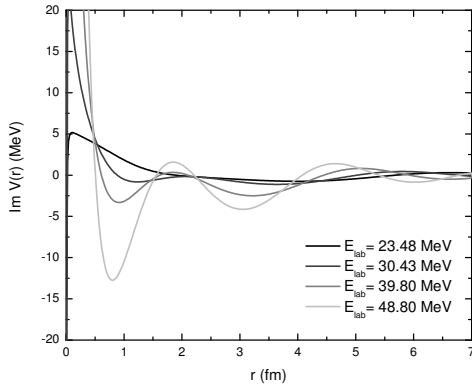
- [5] Münchow M and Scheid W 1980 *Phys. Rev. Lett.* **44** 1299
- [6] May K-E, Münchow M and Scheid W 1984 *Phys. Lett. B* **141** 1
- [7] Apagyi B, Ostrowski A, Scheid W and Voit H 1992 *J. Phys. G: Nucl. Part. Phys.* **18** 195
- [8] Apagyi B, Schmidt A, Scheid W and Voit H 1994 *Phys. Rev. C* **49** 2608
- [9] Alexander N, Amos K, Apagyi B, and Lun D R 1996 *Phys. Rev. C* **53** 88
- [10] Newton R G 1962 *J. Math. Phys.* **3** 75
- [11] Sabatier P C 1966 *J. Math. Phys.* **7** 1515
- [12] Cox J R and Thompson K W 1970 *J. Math. Phys.* **11** 805
- [13] Apagyi B, Harman Z and Scheid W 2003 *J. Phys. A: Math. Theor.* **36** 4815
- [14] Melchert O, Scheid W and Apagyi B 2006 *J. Phys. G: Nucl. Part. Phys.* **32** 849
- [15] Pálmai T, Horváth M and Apagyi B 2008 *J. Phys. A: Math. Theor.* **41** 235305
- [16] Abramowitz M and Stegun I A 1972 *Handbook of Mathematical Functions* (New York: Dover Publications)
- [17] Joachain Ch 1975 *Quantum Collision Theory* (Amsterdam: North-Holland) chapter 17
- [18] Melchert O 2005 *Das Inverse Streuprobem bei fester Energie mit der Methode von Cox und Thompson, Thesis* (University Giessen)
- [19] Thompson I J, Barnett A R (1986) *J. Comput. Phys.* **64** 490
- [20] Dzieciel A, Yngve S (1999) *J. Math. Phys.* **40** 6145
- [21] Ali S, Ahmad A A Z and Ferdous N (1985) *Rev. Mod. Phys.* **57** 923



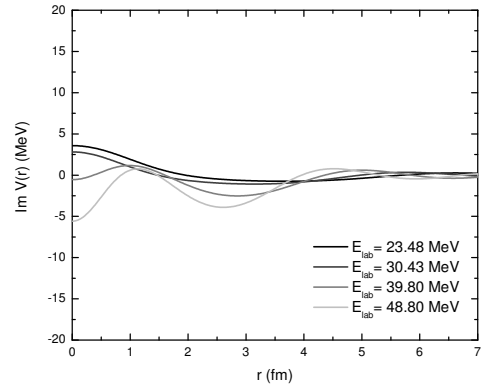
(a) Real part of the CCT potential.



(c) Real part of the PCT potential.



(b) Imaginary part of the CCT potential.



(d) Imaginary part of the PCT potential.

Figure 4: Complex-valued inverse  $p - \alpha$  potentials yielded by the CCT (a,b) and PCT (c,d) methods at various  $E_{\text{lab}}$  proton energies below and above the inelastic threshold using the experimental phase shifts of [21].

- [22] Leeb H, Huber H and Fiedeldey H (1995) *Phys. Lett. B* **344** 18
- [23] Bachelet G B, Hamann D R, Schlüter M (1982) *Phys. Rev. B* **26** 4199
- [24] Gammel J L and Thaler R M (1958) *Phys. Rev.* **109** 2041
- [25] Satchler G R, Owen L W, Elwyn A J, Morgan G L and Walter R L (1968) *Nucl. Phys. A* **112** 1
- [26] Thompson G E, Epstein M B and Sawada T (1970) *Nucl. Phys. A* **142** 571
- [27] Thompson D R, Reichstein I, McClure W and Tang Y C (1969) *Phys. Rev.* **185** 1351